EXISTENCE OF IDENTITIES IN $A \otimes B$

BY

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ABSTRACT

The following theorem is proved: if A and B are two PI-algebras over a field F, then $A \otimes_F B$ is a PI-algebra.

Introduction

Let F be a field, A and B two PI-algebras (algebras satisfying a polynomial identity). The problem whether also $A \otimes_F B$ satisfies a polynomial identity has been open for some time [1, pp. 228]. Procesi and Small [2] had proved that if $B = F_n$ is the algebra of all square matrices of order n over F, then $A \otimes_F F_n \cong A_n$ is a PI-algebra.

In this paper it is proved for arbitrary two PI-algebras A and B, that $A \otimes_F B$ is indeed a PI-algebra. For this purpose we go "back" to the free ring $F[x]$ and the T-ideal Q of identities of a PI-algebra. We define the sequence of co-dimensions $\{d_v\}$ of Q in $F[x]$. A careful study of $\{d_v\}$ shows that T-ideals in $F[x]$ are very large. As an application of the estimation of $\{d_v\}$ we have the theorem which asserts that the tensor product of two PI-algebras is again PI-algebra.

1. Basic notations and definitions

Let F be a field, $\{x\}$ an infinite set of non-commutative indeterminates. Denote the free ring in $\{x\}$ over F by $F[x]$. Let $\{x_n\}$, $\{y_n\} \subseteq \{x\}$ be fixed sequences of indeterminates. We denote by $V_n(x) = Sp\{x_{\sigma_1} \cdots x_{\sigma_n} \mid \sigma \in S_n\}$ the $n!$ dimensional vector space, spanned over F by the $n!$ monomials $x_{\sigma_1} \cdots x_{\sigma_n}, \sigma \in S_n$, where S_n is the group of permutations of $\{1, \cdots, n\}$.

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Similarly, $V_n(y) = Sp\{y_{\sigma_1} \cdots y_{\sigma_n} \mid \sigma \in S_n\}$.

Let $1 \leqq t \leqq n$. Put:

$$V_n^{(t)}(x) = Sp\{x_{\sigma_1} \cdots x_{\sigma_n} = x_t x_{\sigma_2} \cdots x_{\sigma_n} \mid \sigma \in S_n\},$$

i.e, $V_n^{(t)}$ is the $(n-1)!$ dimensional subspace of V_n, spanned by those permutations $\sigma \in S_n$ which starts with $\sigma_1 = t$. It is obvious that $V_n = V_n^{(1)} \oplus V_n^{(2)} \oplus \cdots \oplus V_n^{(n)}$.

Now, let $0 \leqq t \leqq n-1$. We denote by $U_n^{(t)}(x) = V_n^{(t+1)}(x) \oplus V_n^{(t+2)}(x) \oplus \cdots \oplus V_n^{(n)}(x)$ the subspace of $V_n(x)$, spanned by those monomials $x_{\sigma_1} \cdots x_{\sigma_n}$, $\sigma \in S_n$ where $\sigma_1 \notin \{1, \cdots, t\}$. We define also $U_n^{(n)}(x) = (0)$. $U_n^{(t)}(y)$ is defined similarly. It is obvious that $V_n^{(t)} \oplus U_n^{(t)} = U_n^{(t-1)}$, that $V_n = U_n^{(0)}$, and that $U_n^{(1)}(x) = Sp\{x_{\sigma_1} \cdots x_{\sigma_n} \mid \sigma \in S_n \text{ and } \sigma_1 \neq 1\}$.

It is well-known [1, p. 225] that if A is a PI-algebra over a field F, then A satisfies a minimal multilinear homogeneous identity

$$f(x_1, \cdots, x_d) = \sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma_1} \cdots x_{\sigma d}$$

in which the coefficient of the monomial $x_1 \cdots x_d$ is 1. We use this remark as a starting point for our considerations.

Let $3 \leqq d$ and let $f(x_1, \cdots, x_d) \in F[x]$ be a homogeneous multilinear polynomial of degree d, in which the coefficient of $x_1, \cdots x_d$ is 1. We write $f(x_1, \cdots, x_d) = f_d(x_1, \cdots, x_d)$.

DEFINITION 1.1. For any k, $2 \leqq k \leqq d$, let us define (by "decreasing" induction) a polynomial $f_k(x_1, \cdots, x_k)$ as follows:

$$f_d(x_1, \cdots, x_d) = f(x) \text{ has already been defined.}$$

Let $2 \leqq k \leqq d-1$ and assume $f_{k+1}(x_1, \cdots, x_{k+1})$ has been defined. Then gather in $f_{k+1}(x)$ all those monomials which start with x_1 and write:

$$f_{k+1}(x_1, \cdots, x_{k+1}) = x_1 f_k(x_2, \cdots, x_{k+1}) + u_k(x_1, \cdots, x_{k+1})$$

where $u_k(x_1, \cdots, x_{k+1}) \in U_{k+1}^{(1)}(X)$. This relation defines $f_k(x_1, \cdots, x_k)$.

NOTE. The proof (by induction) that the coefficient of the monomial $x_1, \cdots x_k$ in $f_k(x_1, \cdots, x_k)$ is 1, is trivial.

An ideal $Q \subseteq F[x]$ is T-ideal if $g(x_1, \cdots, x_n) \in Q$ and $h_1, \cdots, h_n \in F[x]$ implies that $g(h_1, \cdots, h_n) \in Q$ [1, pp. 233–235].

DEFINITION 1.2. Let $f_2, \cdots, f_d$ be as in Definition 1.1. For $2 \leqq k \leqq d$ we denote the T-ideal generated by $f_k(x_1, \cdots, x_k)$ in $F[x]$ by $P^{(k)}$. We also write

$$P^{(k)} \cap V_n(x) = P_n^{(k)}(x) = P_n^{(k)}.$$

DEFINITION 1.3. Let $2 \leqq k \leqq d$, $0 \leqq l \leqq n$. We define $P_{l,n}^{(k)}$, $P_{l\,n}^{(k)} \subseteq P_n^{(k)}$, to be the sub-vector space of $P_n^{(k)}$ spanned by all polynomials

$$h(x_1, \cdots, x_n) = af_k(g_1, \cdots, g_k)b \text{ which satisfy:}$$

(1) $a, b, g_1, \cdots, g_k$ are monomials in some of the indeterminates $x_1, \cdots, x_n$ and $g_1, \cdots, g_k \neq 1$.

(2) $af_k(g_1, \cdots, g_k)b$ is homogeneous multilinear of degree n in $x_1, \cdots, x_n$.

(3) None of $x_1, \cdots, x_l$ is a left divisor of any of the monomials $g_1, \cdots, g_k$.

If x is a left divisor of a monomial g, we shall also say that g starts with x (from the left). So (3) means that:

$$\text{None of } g_1, \cdots, g_k \text{ starts with any of } x_1, \cdots, x_l.$$

If there is no such $g_1, \cdots, g_k$, we shall write $P_{l\,n}^{(k)} = (0)$. In fact we prove

LEMMA 1.4. *If $n < k + l$, then $P_{l,n}^{(k)} = (0)$.*

PROOF. If $af_k(g_1, \cdots, g_k)b$ is a generator of $P_{l,n}^{(k)}$, then by Definition 1.3, (2), $g_1, \cdots, g_k$ start with k different x_i's. But, by (3) they cannot start with $x_1, \cdots, x_l$. Hence $n - l \geqq k$ or $n \geqq k + l$.

Therefore, if $n < l + k$, there are no generators and $P_{l\,n}^{(k)} = (0)$ Q.E.D.

We note also that it follows from Definition 1.3 (3) that if $l \leqq l'$ then

$$P_{l,\,n}^{(k)} \subseteq P_{l.n}^{(k)}.$$

DEFINITION 1.5. Let H be a T-ideal in $F[x]$ and let $0 < n$ be an arbitrary integer. The integer

$$d_n = \dim_F \frac{V_n(x)}{H \cap V_n(x)}$$

will be called "the co-dimension of order n of H". $\{d_v\}$ is called "the sequence of co-dimensions of H".

DEFINITION 1.6. Let $0 \leqq l_1, \cdots, l_{d-3}$ be any integers. Write

$$W(l_1, \cdots, l_{d-3}, n) = W(l, n) = \sum_{\mu=1}^{d-3} P_{l\,.\,.\,n}^{(d-\mu)}(x) + P_n^{(d)}(x).$$

We define a natural number $a(l_1, \cdots, l_{d-3}, n) = a(l, n)$ as follows:

$$a(l, n) = \dim_F \frac{V_n + W(l, n)}{W(l, n)} = \dim_F \frac{V_n + \sum_{\mu=1}^{d-3} P_{l\,.\,.\,n}^{(d-\mu)} + P_n^{(d)}}{\sum_{\mu=1}^{d-3} P_{l\,.\,.\,n}^{(d-\mu)} + P_n^{(d)}}.$$

For $1 \leqq t \leqq n$ we also define:

$$a^{(t)}(l,n) = \dim_F \frac{V_n^{(t)} + U_n^{(t)} + W(l,n)}{U_n^{(t)} + W(l,n)}.$$

REMARK. The correspondence $x_i \to y_i$, $i = 1,\cdots,n$ induces an isomorphism $\phi: V_n(x) \to V_n(y)$. Since $P^{(k)}$ is a T-ideal, it follows that

$$\phi(P_{l\,n}^{(k)}(x)) = P_{l\,n}^{(k)}(y).$$

Therefore the integers d_n, $a(l,n)$ and $a^{(t)}(l,n)$ are independent of the special sequences $\{x_v\}$ or $\{y_v\}$ used in the definition.

NOTE. We are interested with the co-dimension $d_n = \dim_F \dfrac{V_n + P_n^{(d)}}{P_n^{(d)}}$.

It can be said, roughly, that the integers $a(l_1,\cdots,l_{d-3},n) = a(l,n)$ form a lexicographically ordered way which enable us to pass from the commutative case to that of arbitrary identity of degree d. Note that the proof of Lemma 1.4 implies

$$a(n-2,\cdots,n-2,n) = d_n, \text{ and similarly,}$$

$$a(n,\cdots,n,n) = d_n.$$

Our "guide" in the lexicographic way is

LEMMA 1.7. $$a(l,n) = \sum_{t=1}^{n} a^{(t)}(l,n).$$

PROOF. We have the following chain of vector spaces:

$$V_n = U_n^{(0)} = U_n^{(0)} + W(l,n) \supseteq U_n^{(1)} + W(l,n) \supseteq \cdots \supseteq U_n^{(n)} + W(l,n) = W(l,n).$$

Hence, by using the fact that $U_n^{(t-1)} = V_n^{(t)} \oplus U_n^{(t)}$, we get:

$$a(l,n) = \dim_F \frac{V_n + W(l,n)}{W(l,n)} = \sum_{t=1}^{n} \dim_F \frac{U_n^{(t-1)} + W(l,n)}{U_n^t + W(l,n)}$$

$$= \sum_{t=1}^{n} \dim_F \frac{V_n^{(t)} + U_n^{(t)} + W(l,n)}{U_n^t + W(l,n)} = \sum_{t=1}^{n} a^{(t)}(l,n) \qquad \text{Q.E.D.}$$

REMARK. Let $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$. If $n-2 \leqq l_v$, then $p_{l_v,n}^{(d-v)} = (0)$. To show this, note first that $v \leqq d-3$, hence $3 \leqq d-v$. Since $n-2 \leqq l_v$ we have:

$$n < 3 + n - 2 \leqq d - v + l_v \text{ which,}$$

by Lemma 1.4 implies $P_{l_v,n}^{(d-v)} = (0)$.

Hence, if $n-1 \leqq l_v$, we can replace l_v by $n-2 = h_v$ without changing $P_{v\ n}^{(d-v)} = P_{n-2,n}^{(d-v)} = (0)$. Thus we introduce the following

DEFINITION 1.8. Let $l_0 = 0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$ and write $(l_1, \cdots, l_{d-3}) = l$. We define the sequence $h(l) = (h_1, \cdots, h_{d-3})$ as follows: If $l_{d-3} \leqq n-2$, put $h(l) = l$. If $n - 1 \leqq l_{d-3}$, let ν be the index $(1 \leqq \nu \leqq d-3)$ such that $l_{\nu-1} < n-1 \leqq l_\nu$ and set: $h_1 = l_1, \cdots, h_{\nu-1} = l_{\nu-1},\ h_\nu = \cdots = h_{d-3} = n-2$.

Obviously, $h_1 \leqq \cdots \leqq h_{d-3}$. Moreover we have

PROPOSITION 1.9. *Let $l_0 = 0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$, and let*

$$h = h(l) = (h_1, \cdots, h_{d-3}).$$

Then

$$a(l_1, \cdots, l_{d-3}, n) = a(h_1, \cdots, h_{d-3}, n).$$

PROOF. By Definition 1.6 we have to show that

$$\sum_{\mu=1}^{d-3} P_{l_\mu, n}^{(d-\mu)} = \sum_{\mu=1}^{d-3} P_{h_\mu, n}^{(d-\mu)}.$$

If $l_{d-3} \leqq n-2$ then $l = h(l)$. Suppose $n - 1 \leqq l_{d-3}$ and let $l_{\nu-1} < n-1 \leqq l_\nu$. Then, by Definition 1.7, $h_\mu = l_\mu$, $1 \leqq \mu \leqq \nu - 1$, $h_\mu = n-2$, $\nu \leqq \mu \leqq d-3$. Hence

$$\sum_{\mu=1}^{\nu-1} P_{l_\mu, n}^{(d-\mu)} = \sum_{\mu=1}^{\nu-1} P_{h_\mu, n}^{(d-\mu)}.$$

Moreover, by the preceding remark

$$\sum_{\mu=\nu}^{d-3} P_{l_\mu, n}^{(d-\mu)} = \sum_{\mu=\nu}^{d-3} P_{h_\mu, n}^{(d-\mu)} = (0),$$

hence

$$\sum_{\mu=1}^{d-3} P_{l_\mu, n}^{(d-\mu)} = \sum_{\mu=1}^{d-3} P_{h_\mu, n}^{(d-\mu)} \qquad \text{Q.E.D.}$$

2. Raising commutativity

NOTATION. Let $M(x) = x_{\sigma_1} \cdots x_{\sigma_n} \in V_n(x)$ be a monomial, and let $0 \leqq k \leqq n-1$. $n-k$ of the indices σ_ν satisfy $\sigma_\nu \geqq k+1$, and we denote them by $\mu_1, \mu_2, \cdots, \mu_{n-k}$ according to the order of their appearance in $M(x)$. We denote the other x_{σ_ν}, with $\sigma_\nu \leqq k$, by x_{i_j}, according to their places after the $x_{\mu_j}'s$. With this notation we factorize $M(x)$ into $n-k+1$ blocks which—except the first one—start with some x_{μ_i}:

$$M(x) = (x_{0_1} \cdots x_{0_{r_0}})(x_{\mu_1} x_{1_1} \cdots x_{1_{r_1}}) \cdots (x_{\mu_s} x_{s_1} \cdots x_{s_{r_s}})$$

where $s = n - k$, $\{x_{\mu_1}, \cdots, x_{\mu_s}\} = \{x_{k+1}, \cdots, x_n\}$ and

$$\{x_{0_1}, \cdots, x_{0_{r_0}}, x_{1_1}, \cdots, x_{1_{r_1}}, \cdots, x_{s_1}, \cdots, x_{s_{r_s}}\} = \{x_1, \cdots, x_k\}.$$

Note that $\{0_1, \cdots, 0_{r_0}\}$ might be empty, in which case we shall write $x_{0_1} \cdots x_{0_{r_0}} = 1$ and $M(x) = 1\ (x_{\mu_1} x_{1_1} \cdots x_{1_{r_1}}) \cdots (x_{\mu_s} x_{s_1} \cdots x_{s_{r_s}})$. Note also that some of the r_i's might be zero.

If $k \leqq n-2$, we correspond to $M(x)$ the sequence of $s-1 = n-k-1$ integers $q(M)$:

$$q(M) = (r_1, r_2, \cdots, r_{s-1}).$$

NOTE. The sequence $q(M)$ "measure" the obstruction to $x_{\mu_1}, \cdots, x_{\mu_{n-k}}$ to form a "connected" product $x_{\mu_1} \cdots x_{\mu_{n-k}}$ in $M(x)$. Hence we did not include the ends r_0 and r_s in $q(M)$.

The collection of all such sequences is partially ordered by the lexicographic order:

$(r_1, \cdots, r_{s-1}) < (t_1, \cdots, t_{s-1})$ if there exist $1 \leqq v \leqq s-1$ such that

$r_1 = t_1, \cdots,\ r_{v-1} = t_{v-1}$ and $r_v < t_v$.

THEOREM 2.1. *Let $0 \leqq k \leqq n-1$ and let $M(x) \in V_n(x)$ be a monomial. Then there exists $\delta \in F$ such that $M(x) \equiv \delta N(x) \pmod{P_{k,n}^{(2)}}$ where $N(x)$ is a monomial of the form*

$$(*) \qquad N(x) = x_{\tau_1} \cdots x_{\tau_j} (x_{k+1} \cdots x_n) x_{\tau_{j+1}} \cdots x_{\tau_k},$$

where

$$\{\tau_1, \cdots, \tau_k\} = \{1, \cdots, k\}.$$

PROOF. If $k = n-1$ then $M(x)$ has the (*) form and there is nothing to prove.

Suppose $0 \leqq k \leqq n-2$. Then the proof is divided into two steps. The first step, (*a*), is the reduction of $q(M)$ into $(0) = (0, \cdots, 0)$. The second (*b*), is the reordering of $x_{k+1}, \cdots, x_n$ into their natural order.

(a) Assume $q(M) > (0)$. We prove:

There exist $\alpha \in F$ and a monomial $R(x) \in V_n(x)$ such that $q(R) < q(M)$ and $M(x) \equiv \alpha\ R(x) \pmod{P_{k,n}^{(2)}}$.

Write as before:

$$M(x) = (x_{0_1} \cdots x_{0_{r_0}}) \cdots (x_{\mu_s} x_{s_1} \cdots x_{s_{r_s}}),\ q(M) = (r_1, \cdots, r_{s-1}),\ s = n-k.$$

Since $0 < q(M)$, there exists $1 \leqq j \leqq s-1$ such that $q(M) = (r_1, \cdots, r_j, 0, \cdots, 0)$, $r_j \neq 0$. Let $a(x) = (x_{0_1} \cdots x_{0_{r_0}}) \cdots (x_{\mu_{j-1}} x_{j-1_1} \cdots x_{j-1_{r_{j-1}}})$ be the first j blocks in $M(x)$, and let

$$b(x) = x_{\mu_{j+2}} x_{\mu_{j+3}} \cdots x_{\mu_{s-1}} (x_{\mu_s} x_{s_1} \cdots x_{s_{r_s}}) \text{ be the last } s-j-1 \text{ blocks.}$$

Notice that if $j = 1$, then $a(x) = x_{0_1} \cdots x_{0_{r_0}}$. If $r_0 = 0$, $a(x) = 1$. Note also that if $j = s-1$, then $b(x) = x_{s_1} \cdots x_{s_{r_s}}$.

We can now write:

$$M(x) = a(x)(x_{\mu_j} x_{j_1} \cdots x_{j_{r_j}}) x_{\mu_{j+1}} b(x).$$

Let $R(x) = a(x) x_{\mu_{j+1}} (x_{\mu} x_{j_1} \cdots x_{j_{r_j}})\ b(x)$ be the monomial derived from $M(x)$ by permuting the two blocks $g_1 = x_{\mu_j} x_{j_1} \cdots x_{j_{r_j}}$ and $g_2 = x_{\mu_{j+1}}$.

Obviously, $q(R) = (r_1, \cdots, r_{j-1}, 0, r_j, 0, \cdots, 0)$. (If $j = s-1$ then $q(R) = (r_1, \cdots, r_{s-2}, 0)$). Hence $q(R) < q(M)$.

Now, $f_2(u,v) = uv - \alpha v u$. Therefore we can write:

$$M(x) - \alpha R(x) = a(x) f_2(x_{\mu_j} x_{j_1} \cdots x_{j_{r_j}}, x_{\mu_{j+1}}) b(x).$$

$g_1 = u_{\mu_j} \cdots x_{j_{r_j}}$ and $g_2 = x_{\mu_{j+1}}$ starts with x_{μ_j} and $x_{\mu_{j+1}}$.

Since μ_j, $\mu_{j+1} > k$, g_1 and g_2 do not start with any of $x_1, \cdots, x_k$, and therefore $M(x) - \alpha R(x) = a(x) f_2(g_1, g_2)\, b(x) \in P_{k,n}^{(2)}$.

We continue this procedure successively on $R(x)$ until we have $\beta \in F$ and a monomial $S(x) \in V_n(x)$ such that $M(x) \equiv \beta S(x) (\mathrm{mod}\, P_{k,n}^{(2)})$ and $q(S) = (0)$. Therefore

$$S(x) = x_{\tau_1} \cdots x_{\tau_j} (x_{\rho_1} \cdots x_{\rho_{n-k}}) x_{\tau_{j+1}} \cdots x_{\tau_k} \text{ where } \{\tau_1, \cdots, \tau_k\} = \{1, \cdots, k\}$$

and $\{\rho_1, \cdots, \rho_{n-k}\} = \{k+1, \cdots, n\}$.

(b) Let $S(x)$ be as above and let $T(x) = x_{\tau_1} \cdots x_{\tau_j} (x_{\rho_1} \cdots x_{\rho_{v+1}} x_{\rho_v} \cdots x_{\sigma_{v-k}}) x_{\tau_{j+1}} \cdots x_{\tau_k}$ be a monomial derived from $S(x)$ by permuting two neighbouring variables in the block $x_{\rho_1} \cdots x_{\rho_{n-k}}$. We shall show that $S(x) \equiv \alpha\ T(x)\ (\mathrm{mod}\, P_{k,n}^{(2)})$. The proof is similar to that of part (a):

Write:

$$a(x) = x_{\tau_1} \cdots x_{\tau_j} x_{\rho_1} \cdots x_{\rho_{v-1}},$$

$$b(x) = x_{\rho_{v+2}} \cdots x_{\rho_{n-k}} x_{\tau_{j+1}} \cdots x_{\tau_k}.$$

Then: $S(x) - \alpha \cdot T(x) = a(x) f_2(x_{\rho_v}, x_{\rho_{v+1}}) b(x) \in P_{k,n}^{(2)}$
and $S(x) \equiv \alpha T(x)\ (\mathrm{mod}\ P_{k,n}^{(2)})$.

We apply this procedure in succession and reorder $x_{\rho_1}, \cdots, x_{\rho_{n-k}}$ to their natural order $x_{k+1} \cdots x_n$, so that we have $\gamma \in F$ such that $S(x) \equiv \gamma\ N(x)\ (\mathrm{mod}\ P_{k,n}^{(2)})$ where

$$N(x) = x_{\tau_1} \cdots x_{\tau_j} (x_{k+1} \cdots x_n) x_{\tau_{j+1}} \cdots x_{\tau_k}.$$

Combining (a) and (b), we conclude that there exists $\delta \in F$ such that $M(x) = \delta\ N(x)\ (\mathrm{mod}\ P_{k,n}^{(2)})$ with $N(x)$ as above. Q.E.D.

Let $0 \leqq k \leqq n-1$ and let θ denote the one-to-one linear transformation induced by the correspondence: $y_1 \to x_1, \cdots, y_k \to x_k$ and $y_{k+1} \to (x_{k+1} \cdots x_n)$: $\theta\colon V_{k+1}(y) \to V_n(x)$. We can now re-write the preceding theorem in the following way:

THEOREM 2.2. *Let* $0 \leqq k \leqq n-1$, *then*

$$V_n(x) = \theta(V_{k+1}(y)) + P^{(2)}_{k,n}(x).$$

PROOF. Obviously, $V_n(x) \supseteq \theta(V_{k+1}(y)) + P^{(2)}_{k,n}(x)$.

The inverse inclusion follows from Theorem 1.1 and the fact that

$$N(x) = x_{\tau_1} \cdots x_{\tau}\,(x_{k+1} \cdots x_n) x_{\tau_{j+1}} \cdots x_{\tau_k} \in \theta(V_{k+1}(y)). \qquad \text{Q.E.D.}$$

3. Recursive estimation for $a(l, n)$

Let $1 \leqq t \leqq n$. We shall use the short notation

$$(\hat{x}_t) = (x_1, \cdots, \hat{x}_t, \cdots, x_n) = (x_1, \cdots, x_{t-1}, x_{t+1}, \cdots, x_n).$$

Let $V_{n-1}(\hat{x}_t) = Sp\{x_{\sigma_1} \cdots x_{\sigma_{n-1}} \mid \{\sigma_1, \cdots, \sigma_{n-1}\} = \{1, \cdots, t-1,\ t+1, \cdots n\}\}$ denote the $(n-1)!$ space spanned by all $(n-1)!$ multilinear monomials in $(\hat{x}_t)$. Let ϕ be the isomorphism induced by the correspondence

$$y_1 \to x_1, \cdots, y_{t-1} \to x_{t-1}, y_t \to x_{t+1}, \cdots, y_{n-1} \to x_n.$$

Write:

$$\phi(P^{(k)}_{l\,n-1}(y)) = P^{(k)}_{l,n-1}(\hat{x}_t),\ \ \phi(P^{(d)}_{n-1}(y)) = P^{(d)}_{n-1}(\hat{x}_t).$$

It can be shown that $P^{(k)}_{l,n-1}(\hat{x}_t)$ is the subspace of $V_{n-1}(\hat{x}_t)$, generated by 0 and by all polynomials $af_k(g_1, \cdots, g_k)b$ where:

(1′) $a, b, g_1, \cdots, g_k$ are monomial in $(\hat{x}_t)$ and $g_1, \cdots, g_k \neq 1$.

(2′) $af_k(g_1, \cdots, g_k)b$ is homogeneous multilinear in $(\hat{x}_t)$.

(3′) None of the g_i's starts with any of the first l x_i's from the sequence $x_1, \cdots, \hat{x}_t, \cdots, x_n$.

NOTE. these first l $x'_i s$ are $x_1, \cdots, x_l$ in case $l < t$, and $x_1, \cdots, x_{t-1}, x_{t+1}, \cdots, x_{l+1}$ if $t \leqq l$.

Since ϕ is an isomorphism, we can write for $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n-1$.

$$a(l, n-1) = \dim_F \frac{V_{n-1}(\hat{x}_t) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu, n-1}(\hat{x}_t) + P^{(d)}_{n-1}(\hat{x}_t)}{\sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu, n-1}(\hat{x}_t) + P^{(d)}_{n-1}(\hat{x}_t)}.$$

$V_n^{(t)}(x)$ is the subspace spanned by all monomials $x_{\sigma_1} \cdots x_{\sigma_n} = x_t x_{\sigma_2} \cdots x_\sigma$ where $\sigma_1 = t$. Hence $(\sigma_2, \cdots, \sigma_n)$ is a permutation of $(1, \cdots, \hat{t}, \cdots, n)$, therefore

$$x_{\sigma_2} \cdots x_{\sigma_n} \in V_{n-1}(\hat{x}_t). \text{ Hence } V_n^{(t)}(x) = x_t V_{n-1}(\hat{x}_t).$$

For the spaces $P_{l\,n-1}^{(k)}(\hat{x}_t)$ and $P_{n-1}^{(d)}(\hat{x}_t)$ we have

LEMMA 3.1. *Let $0 \leqq k, l, n$ be any integers and let $1 \leqq t \leqq n$. Then*

(i) $$x_t P_{l\,n-1}^{(k)}(\hat{x}_t) \subseteq P_{l\,n}^{(k)}(x) \text{ and}$$

$$x_t P_{n-1}^{(d)}(\hat{x}_t) \subseteq P_n^{(d)}(x)$$

(ii) $$x_t P_{t-1,n-1}^{(k)}(\hat{x}_t) \subseteq P_{t-1,n}^{(k+1)}(x) + U_n^{(t)}(x).$$

PROOF. (i) If $P_{l,n-1}^{(k)}(\hat{x}_t) = (0)$, there is nothing to prove. Hence assume that $af_k(g_1, \cdots, g_k)b \in P_{l,n-1}^{(k)}(\hat{x}_i)$ is a generator. We show that $x_i af_k(g_1, \cdots, g_k)b \in P_{l\,n}^k(x)$. Clearly $x_t af_k(g_1, \cdots, g_k)b$ is homogeneous multilinear in $x_1, \cdots, x_n$. $x_t a, b, g_1, \cdots, g_k$ are monomials and $g_1, \cdots, g_k \neq 1$. Therefore we have to show that $g_1, \cdots, g_k$ do not start with $x_1, \cdots, x_l$. It is given that $g_1, \cdots, g_k$ do not start with the first l x_i's from $(x_1, \cdots, \hat{x}_t, \cdots, x_n)$. If $l < t$, these are $x_1, \cdots, x_l$—which was to be proved. If $t \leqq l$, these are $x_1, \cdots, \hat{x}_t, \cdots, x_{l+1}$. But x_t does not appear at all in the g_i's, hence they do not start with $x_1, \cdots, x_{l+1}$ and, in particular, not with $x_1, \cdots, x_l$.

The inclusion $x_t P_{l,n-1}^{(k)}(\hat{x}_t) \subseteq P_{l\,n}^{(k)}(x)$ follows by similar arguments.

(ii) Suppose $P_{t-1,n-1}^{(k)}(\hat{x}_t) \neq (0)$ and let $af_k(g_1, \cdots, g_k)b \in P_{t-1\,n-1}^{(k)}(\hat{x}_t)$ be one of the generators. As before, this means that the g_i's are monomials $\neq 1$ in $x_1, \cdots, \hat{x}_t, \cdots, x_n$, which do not start with $x_1, \cdots, x_{t-1}$. Since x_t does not appear in the g_i's, $g_1, \cdots, g_k$ can be considered as monomials in $x_1, \cdots, x_n$ which do not start with $x_1, \cdots, x_t$.

By Definition 1.1 we have for any non-commutative indeterminates

$$y\text{'s: } f_{k+1}(y_1, \cdots, y_{k+1})y = y_1 f_k(y_2, \cdots, y_{k+1})y + u_k(y_1, \cdots, y_{k+1})y$$

where $u_k(y_1, \cdots, y_{k+1}) \in U_{k+1}^{(1)}(y)$. Specialize now $(y_1, \cdots, y_{k+1}, y) \to (x_t a, g_1, \cdots, g_k, b)$ to have the equality

$$f_{k+1}(x_i a, g_1, \cdots, g_k)b = x_i af_k(g_1, \cdots, g_k)b + u_k(x_t a, g_1, \cdots, g_k, b).$$

Now, $u_k(y_1, \cdots, y_{k+1}) \in U_{k+1}^{(1)}(y)$ implies that $u_k(y_1, \cdots, y_{k+1})$ is a sum of monomials in $y_1, \cdots, y_{k+1}$, none of which starts with y_1, hence it starts with some y_j, $2 \leqq j$. Therefore $u_k(x_t a, g_1, \cdots, g_k)$ is a sum of monomials, each of which starts with one of the g_i's. Since $g_1, \cdots, g_k \neq 1$ and they do not start with $x_1, \cdots, x_t$, it follows that none of the monomials of $u_k(x_ta, g_1, \cdots, g_k)b$ starts with any of $x_1, \cdots, x_t$. Since obviously $u_k(x_ta, g_1, \cdots, g_k)b \in V_n(x)$ we have that $u_k(x_ta, g_1, \cdots, g_k)b \in U_n^{(t)}(x)$.

To show that $f_{k+1}(x_ta, g_1, \cdots, g_k)b \in P_{t-1,n}^{(k+1)}(x)$, notice that it is a multilinear homogeneous polynomial in $x_1, \cdots, x_n$, that $1, b, x_ta, g_1, \cdots, g_k$ are monomials and that $x_ta, g_1, \cdots, g_k \neq 1$. Hence we have only to show that none of $x_ta, g_1, \cdots, g_k$ starts with any of $x_1, \cdots, x_{t-1}$. This is obvious since we are given that the g_i's do not start with $x_1, \cdots, x_t$ and x_ta do not start with $x_1, \cdots, x_{t-1}$.

Finally we have:

$$x_taf_k(g_1, \cdots, g_k)b = f_{k+1}(x_ta, g_1, \cdots, g_k)b - u_k(x_ta, g_1, \cdots, g_k)b \in$$
$$\in P_{t-1,n}^{(k+1)}(x) + U_n^{(t)}(x)$$

Q.E.D.

In the following Lemmas 3.2, 3.4 and 3.5 we shall constantly use the following trivial:

REMARK. Let $C \subseteq B \subseteq A$ be three finite dimensional vector spaces. Then

$$\dim \frac{A+B}{B} \leqq \dim \frac{A+C}{C}$$

LEMMA 3.2. *Let* $l_0 = 0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$ *and let* $l_{v-1} \leqq t$ *for some* $1 \leqq v \leqq d-3$. *Then*

$$a^{(t)}(l_1, \cdots, l_v, \cdots, l_{d-3}, n) \leqq a(l_1, \cdots, l_{v-1}, t-1, l_{v+1}, \cdots, l_{d-3}, n-1)$$

PROOF. Since $x_tV_{n-1}(\hat{x}_t) = V_n^{(t)}(x)$, it is obvious that left multiplication by x_t induces isomophorphism of $V_{n-1}(\hat{x}_t)$ onto $V_n^{(t)}(x)$. Use this isomorphism to deduce the following equalities:

$$a = a(l_1, \cdots, l_{v+1}, \cdots, l_{d-3}, n-1)$$
$$= \dim_F \frac{V_{n-1}(\hat{x}_t) + \sum_{\mu \neq v} P_{l_\mu,n-1}^{(d-\mu)}(\hat{x}_t) + P_{t-1,n-1}^{(d-v)}(\hat{x}_t) + P_{n-1}^{(d)}(\hat{x}_t)}{\sum_{\mu \neq v} P_{l_\mu,n-1}^{(d-\mu)}(\hat{x}_t) + P_{t-1,n-1}^{(d-v)}(\hat{x}_t) + P_{n-1}^{(d)}(\hat{x}_t)}$$
$$= \dim_F \frac{V_n^{(t)}(x) + \sum_{\mu \neq v} x_tP_{l_\mu,n-1}^{(d-\mu)}(\hat{x}_t) + x_tP_{t-1,n-1}^{(d-v)}(\hat{x}_t) + x_tP_{n-1}^{(d)}(\hat{x}_t)}{\sum_{\mu \neq v} x_tP_{l_\mu,n-1}^{(d-\mu)}(\hat{x}_t) + x_tP_{t-1,n-1}^{(d-v)}(\hat{x}_t) + x_tP_{n-1}^{(d)}(\hat{x}_t)} \quad \cdots(*)$$

By Lemma 3.1. we have:

(i) $$\sum_{\mu \neq v} x_tP_{l_\mu,n-1}^{(d-\mu)}(\hat{x}_t) \subseteq \sum_{\mu \neq v} P_{l_\mu,n}^{(d-\mu)}(x) \text{ and}$$

$$x_tP_{n-1}^{(d)}(\hat{x}_t) \subseteq P_n^{(d)}(x)$$

(ii) $$x_t P^{(d-\nu)}_{t-1\, n-1}(\hat{x}_t) \subseteq P^{(d-\nu+1)}_{t-1,n}(x) + U^{(t)}_n(x).$$

By assumption, $l_{\nu-1} \leqq t-1$, hence:

(iii) $$P^{(d-(\nu-1))}_{t-1,n}(x) \subseteq P^{(d-(\nu-1))}_{l_{\nu-1},n}(x) \subseteq \sum_{\mu \neq \nu} P^{(d-\mu)}_{l_\mu,n}(x).$$

Therefore:

$$x_t P^{(d-\nu)}_{t-1,n-1}(\hat{x}_t) \subseteq \sum_{\mu \neq \nu} P^{(d-\mu)}_{l_\nu,n}(x) + U^{(t)}_n(x) \subseteq \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n} + U^{(t)}_n.$$

Now, (i)—(iii) imply that the denominator in (*) is contained in

$$\sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n} + P^{(d)}_n + U^{(t)}_n.$$

Hence it follows by the preceding remark that

$$a \geqq \dim_F \frac{V^{(t)}_n + U^{(t)}_n + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n} + P^{(d)}_n}{U^{(t)}_n + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu\, n} + P^{(d)}_n} = a^{(t)}(l_1, \cdots, l_{d-3}, n).$$

Q.E.D.

Let $t \leqq n-1$. The correspondence $y_1 \to x_1, \cdots, y_{t-1} \to x_{t-1},\ y_t \to (x_{t+1} \cdots x_n)$ induces a monomorphism $\xi: V_t(y) \to V_{n-1}(\hat{x}_t)$ of $V_t(y)$ into $V_{n-1}(\hat{x}_t)$. Let $\eta: V_{n-1}(\hat{x}_t) \to V^{(t)}_n(x)$ denote the isomorphism obtained by left multiplication by x_t, and let $\psi = \eta o \xi$. With this notation we have

LEMMA 3.3.

(1) $V^{(t)}_n(x) \subseteq \psi(V_t(y)) + P^{(3)}_{t-1,n}(x) + U^{(t)}_n(x)$

(2) $\psi(P^{(k)}_{l,t}(y)) \subseteq P^{(k)}_{l,n}(x)$ and $\psi(P^{(k)}_t(y)) \subseteq P^{(k)}_n(x)$.

PROOF. (1) Theorem 2.2 asserts that

$$V_n(x) = \theta(V_{k+1}(y)) + P_{k,n}(x).$$

Take $n-1$ instead of n, $t = k+1$ and ξ instead of θ, to conclude that

$$V_{n-1}(\hat{x}_t) = \xi(V_t(y)) + P^{(2)}_{t-1\, n-1}(\hat{x}_t).$$

Multiply both sides by x_t and use Lemma 3.1, (ii) to get

$$V^{(t)}_n(x) \subseteq \psi(V_t(y)) + P^{(3)}_{t-1,n}(x) + U^{(t)}_n(x).$$

(2) Assume $P^{(k)}_{l,t}(y) \neq (0)$ and let $af_k(g_1, \cdots, g_k)b$ be one of its generators. We show that

$$\psi(af_k(g_1,\cdots,g_k)b) = x_t\xi(a)f_k(\xi(g_1),\cdots,\xi(g_k))\xi(b) \in P_{l,n}^{(k)}(x).$$

Denote $\bar{a} = \xi(a)$, $\bar{b} = \xi(b)$, $\bar{g}_i = \xi(g_i)$. It is obvious that $x_t\bar{a}f_k(\bar{g}_1,\cdots,\bar{g}_k)\bar{b} \in P_n^{(k)}(x)$ and $\bar{g}_1,\cdots,\bar{g}_k \neq 1$. Hence we have to show that none of $\bar{g}_1,\cdots,\bar{g}_k$ starts with any of $x_1,\cdots,x_l$.

By assumption, $P_{l,t}^{(k)}(y) \neq (0)$, hence $l + k \leqq t$. Since $2 \leqq k$, $l < t$. Hence $\xi: (y_1,\cdots,y_l) \to (x_1,\cdots,x_l)$.

Since the g_i's do not start with $y_1,\cdots,y_l$, it follows that the $\bar{g}_i$'s do not start with $x_1,\cdots,x_l$, which was to be proved.

The inclusion $\psi(P_t^{(k)}(y)) \subseteq P_n^{(k)}(x)$ is obvious. Q.E.D.

LEMMA 3.4. *Let* $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} < t \leqq n-1$, *then*

$$a^{(t)}(l_1,\cdots,l_{d-3},n) \leqq a(l_1,\cdots,l_{d-3},t).$$

PROOF. Let $\psi = \eta \circ \xi$ be the above monomorphism. By Lemma 3.3 and the Remark which preceded Lemma 3.2 we have: $a = a(l,t)$

$$= \dim_F \frac{V_t(y) + \sum_{\mu=1}^{d-3} P_{l_\mu,t}^{(d-\mu)}(y) + P_t^{(d)}(y)}{\sum_{\mu=1}^{d-3} P_{l_\mu,t}^{(d-\mu)}(y) + P_t^{(d)}(y)}$$

$$= \dim_F \frac{\psi(V_t(y)) + \sum_{\mu=1}^{d-3} \psi(P_{l_\mu,t}^{(d-\mu)}(y)) + \psi(P_t^{(d)}(y))}{\sum_{\mu=1}^{d-3} \psi(P_{l_\mu,t}^{(d-\mu)}(y)) + \psi(P_t^{(d)}(y))}$$

$$\geqq \dim_F \frac{\psi(V_t(y)) + \sum_{\mu=1}^{d-3} P_{l_\mu,n}^{(d-\mu)}(x) + P_n^{(d)}(x)}{\sum_{\mu=1}^{d-3} P_{l_\mu,n}^{(d-\mu)}(x) + P_n^{(d)}(x)} = b.$$

Adding $U_n^{(t)}(x)$ to both nominator and denominator and using the above Remark we have

$$b \geqq \dim_F \frac{\psi(V_t(y)) + U_n^{(t)}(x) + \sum_{\mu=1}^{d-3} P_{l_\mu,n}^{(d-\mu)}(x) + P_n^{(d)}(x)}{U_n^{(t)}(x) + \sum_{\mu=1}^{d-3} P_{l_\mu,n}^{(d-\mu)}(x) + P_n^{(d)}(x)} = c.$$

Now, $l_{d-3} \leqq t-1$, hence $P_{t-1,n}^{(3)} P \leqq {}_{l_{d-3},n}^{(3)}$, so that adding $P_{t-1,n}^{(3)}(x) + U_n^{(t)}(x)$ to the nominator does not change it. Therefore:

$$c = \dim_F \frac{[\psi(V_t(y)) + P^{(3)}_{t-1,n}(x) + U^{(t)}_n(x)] + U^{(t)}_n + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P^{(d)}_n(x)}{U^{(t)}_n + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P^{(d)}_n(x)} \geqq$$

(by Lemma 3.3, (1))

$$\geqq \dim_F \frac{V^{(t)}_n(x) + U^{(t)}_n(x) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P^{(d)}_n(x)}{U^{(t)}_n(x) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P^{(d)}_n(x)}$$

$$= a^{(t)}(l_1, \cdots, l_{d-3}, n)$$ Q.E.D.

Let $V_{n-1} = V_{n-1}(x_1, \cdots, x_{n-1}) = V_{n-1}(\hat{x}_n)$. Clearly, left multiplication by x_n induces an isomorphism of V_{n-1} onto $V_n^{(n)} = x_n V_{n-1}(\hat{x}_n)$. By Lemma 3.1, (i), $x_n P^{(k)}_{l\,n-1}(\hat{x}_n) \subseteq P^{(k)}_{l,n}(x)$ and $x_n P^{(d)}_{n-1}(\hat{x}_n) \subseteq P_n^{(d)}$. Using these remarks we can prove:

LEMMA 3.5. *Let* $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$, *then* $a^{(n)}(l_1, \cdots, l_{d-3}, n) \leqq a(l_1, \cdots, l_{d-3}, n-1)$.

PROOF.

$$a(l, n-1) = \dim_F \frac{V_{n-1}(\hat{x}_n) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n-1}(\hat{x}_n) + P^{(d)}_{n-1}(\hat{x}_n)}{\sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n-1}(\hat{x}_n) + P^{(d)}_{n-1}(\hat{x}_n)}.$$

Left multiplication by x_n implies

$$a(l, n-1) = \dim_F \frac{V^{(n)}_n(x) + \sum_{\mu=1}^{d-3} x_n P^{(d-\mu)}_{l_\mu,n-1}(\hat{x}_n) + x_n P^{(d)}_{n-1}(\hat{x}_n)}{\sum_{\mu=1}^{d-3} x_n P^{(d-\mu)}_{l_\mu,n-1}(\hat{x}_n) + x_n P^{(d)}_{n-1}(\hat{x}_n)} = a$$

Use the preceding remarks and the fact that $U_n^{(n)} = (0)$, to obtain that the denominator of a is contained in $U_n^{(n)}(x) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P_n^{(d)}(x)$. Hence

$$a(l, n-1) \geqq \dim_F \frac{V^{(n)}_n(x) + U^{(n)}_n(x) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P^{(d)}_n(x)}{U^{(n)}_n(x) + \sum_{\mu=1}^{d-3} P^{(d-\mu)}_{l_\mu,n}(x) + P^{(d)}_n(x)}$$

$$= a^{(n)}(l, n)$$ Q.E.D.

As a corollary to Lemmas 3.2, 3.4 and 3.5 we have the following recursive estimation for the integers $a(l,n)$:

THEOREM 3.6. *Let* $l_0 = 0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n-2$. *Then*

$$a(l_1,\cdots,l_{d-3},n)$$
$$\leqq \sum_{\nu=1}^{d-3} \sum_{t=l_{\nu-1}+1}^{l_\nu} a(l_1,\cdots,l_{\nu-1},t-1,l_{\nu+1},\cdots,l_{d-3},n-1)$$
$$+ \sum_{t=l_{d-3}+1}^{n-2} a(l_1,\cdots,l_{d-3},t) + 2a(l_1,\cdots,l_{d-3},n-1) \cdots (**)$$

REMARK. If $l_{\nu-1} = l_\nu$ then the summand $\sum_{t=l_{\nu-1}+1}^{l_\nu} a(l_1,\cdots,l_{\nu-1}, t-1, l_{\nu+1},\cdots,l_{d-3},n-1)$ is empty and equal zero. In the same way, if $l_{d-3} = n-2$, then $\sum_{t=l_{d-3}+1}^{n-2} a(l_1,\cdots,l_{d-3},t) = 0$. Note also that each summand in the right side of the inequality is of the form $a(s_1,\cdots,s_{d-3},k)$ where $0 \leqq s_1 \leqq \cdots \leqq s_{d-3} \leqq k \leqq n-1$.

PROOF OF THEOREM 3.6. By Lemma 1.6

$$a(l,n) = \sum_{t=1}^{n} a^{(t)}(l,n)$$
$$= \sum_{\nu=1}^{d-3} \sum_{t=l_{\nu-1}+1}^{l_\nu} a^{(t)}(l,n) + \sum_{t=l_{d-3}+1}^{n} a^{(t)}(l,n) \cdots (1).$$

In the summand $\sum_{t=l_{\nu-1}+1}^{l_\nu} a^{(t)}(l,n)$ we can apply Lemma 3.2 for $l_{\nu-1} < t \leqq l_\nu$ to get:

$$\sum_{t=l_{\nu-1}+1}^{l_\nu} a^{(t)}(l,n) \leqq \sum_{t=l_{\nu-1}+1}^{l_\nu} a(l_1,\cdots,l_{\nu-1},t-1,l_{\nu+1},\cdots,l_{d-3},n-1)$$

so that:

$$\sum_{\nu=1}^{d-3} \sum_{t=l_{\nu-1}+1}^{l_\nu} a^{(t)}(l,n) \leqq \sum_{\nu=1}^{d-3} \sum_{t=l_{\nu-1}+1}^{l_\nu} a(l_1,\cdots,l_{\nu-1},t-1,l_{\nu+1},\cdots,l_{d-3},n-1).$$

Write:

$$\sum_{t=l_{d-3}+1}^{n} a^{(t)}(l,n) = \sum_{t=l_{d-3}+1}^{n-1} a^{(t)}(l,n) + a^{(n)}(l,n).$$

For $l_{d-3} < t \leqq n-1$ we apply Lemma 3.4 so that

$$\sum_{t=l_{d-3}+1}^{n-1} a^{(t)}(l,n) \leqq \sum_{t=l_{d-3}+1}^{n-1} a(l,t) = \sum_{t=l_{d-3}+1}^{n-2} a(l,t) + a(l,n-1).$$

Note that since we assume $l_{d-3} \leqq n-2$, the first summand may be empty, but the last summand $a(l,n-1)$ will always appear.

Finally, we use Lemma 3.5 to obtain $a^{(n)}(l,n) \leqq a(l,n-1)$. It follows therefore that

$$\sum_{t=l_{d-3}+1}^{n} a^{(t)}(l,n) \leqq \sum_{t=l_{d-3}+1}^{n-2} a(l,t) + 2a(l,n-1) \quad \cdots \quad (3)$$

The Theorem now follows from (1), (2) and (3) Q.E.D.

4. Numerical estimation for $a(l,n)$

Since $\dim_F V_1 = 1$, it is obvious that for any integers $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$ we have $a(l,1) \leqq 1$. In particular, $a(0,\cdots,0,1) \leqq 1$.

In order to be able to get an estimation for the ingers $a(l,n)$ we define by induction another set of integers $A(l,n)$, for which we shall be able to estimate (**) of Theorem 3.6.

This is done as follows:

DEFINITION 4.1. Let $1 \leqq n$ and let $l_0 = 0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$. Define by induction on n the integers $A(l_1,\cdots,l_{d-3},n)$ as follows:

1) Let $n=1$ and let $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq 1$. Define $A(l_1,\cdots,l_{d-3},1) = 1$.

2) Assume that for any $t \leqq n-1$ and a sequence $0 \leqq s_1 \leqq \cdots \leqq s_{d-3} \leqq t$, the integers $A(s_1,\cdots,s_{d-3},t)$ have been defined. Let $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$ and define $A(l_1,\cdots,l_{d-3},n)$ as follows:

Case I. Let $l_{d-3} \leqq n-2$. Put

$$A(l_1,\cdots,l_{d-3},n) = \sum_{\nu=1}^{d-3} \sum_{t=l_{\nu-1}+1}^{l_\nu} A(l_1,\cdots,l_{\nu-1},t-1,l_{\nu+1},\cdots,l_{d-3},n-1)$$
$$+ \sum_{t=l_{d-3}+1}^{n-2} A(l_1,\cdots,l_{d-3},t) + 2A(l_1,\cdots,l_{d-3},n-1)$$

and note that each term is given by induction. Note also that some of the summands $\sum_{t=l_{\nu-1}+1}^{l_\nu} A(l_1,\cdots,t-1,\cdots,l_{d-3},n-1)$ and $\sum_{t=l_{d-3}+1}^{l_\nu} A(l_1,\cdots,l_{d-3},t)$ may equal zero.

Case II. Let $l_{d-3} \geqq n-1$. Let $h(l) = (h_1,\cdots,h_{d-3})$ be the sequence defined in Definition 1.8. $A(h_1,\cdots,h_{d-3},n)$ is now defined by case I and we set $A(l_1,\cdots,l_{d-3},n) = A(h_1,\cdots,h_{d-3},n)$.

Estimation for $A(l,n)$ is indeed an estimation for $a(l,n)$ because of

PROPOSITION 4.2. *Let* $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$ *then* $a(l,n) \leqq A(l,n)$.

PROOF. By induction on n. If $n=1$, $a(l,1) \leqq 1 = A(l,1)$.

Assume the proposition is valid for all sequences $0 \leqq s_1 \leqq \cdots \leqq s_{d-3} \leqq t < n$. If $l_{d-3} \leqq n-2$, use the inductive definition, case I—of $A(l,n)$—and apply Theorem 3.6 to obtain $a(l,n) \leqq A(l,n)$.

If $l_{d-3} \geqq n-1$, use the sequence $h(l)$. By Proposition 1.9, $a(h(l),n) = a(l,n)$ Since $h_{d-3} \leqq n-2$, it follows now that $a(h(l),n) \leqq A(h(l),n)$. Hence:

$$a(l,n) = a(h(l),n) \leqq A(h(l),m) = A(l,n). \qquad \text{Q.E.D.}$$

LEMMA 4.3. *Let* $l_0 = 0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n-2$, *then*

$$A(l,n) \leqq 3\left[\sum_{v=1}^{d-4} \sum_{t=l_{v-1}+1}^{l_v} A(l_1,\cdots,l_{v-1},t-1,l_{v+1},\cdots,l_{d-3},n-1) + \sum_{t=l_{d-4}+1}^{l_{d-3}+1} A(l_1,\cdots,l_{d-4},t-1,n-1)\right] = B \quad \cdots (*)$$

PROOF. Denote the right side of (*) by B, and note that B can also be written as follows:

$$B = 3\left[\sum_{v=1}^{d-3} \sum_{t=l_{v-1}+1}^{l_v} A(l_1,\cdots,t-1,\cdots,l_{d-3},n-1) + A(l_1,\cdots,l_{d-3},n-1)\right].$$

We prove that $A(l,n) \leqq B$, using the second form of B.

Since $l_{d-3} \leqq n-2$, we can use Case I of Definition 4.1, so that

$$A(l,n) = \sum_{v=1}^{d-3} \sum_{t=l_{v-1}+1}^{l_v} A(l_1,\cdots,l_{v-1},t-1,l_{v+1},\cdots,l_{d-3},n-1) + \sum_{t=l_{d-3}+1}^{n-2} A(l_1,\cdots,l_{d-3},t) + 2A(l_1,\cdots,l_{d-3},n-1) = \alpha + \beta + 2\gamma.$$

If $l_{d-3} = n-2$, the second summand β is zero: $\beta = \sum_{t=l_{d-3}+1}^{n-2} A(l,t) = 0$, so that $A(l,n) = \alpha + 2\gamma \leqq 3(\alpha+\gamma)$ which was to be proved.

If $l_{d-3} \leqq n-3$, the integer $\gamma = A(l,n-1)$ is also given by Definition 4.1, Case I—by substituting $n-1$ for n. Hence

$$\gamma = A(l,n-1) = \sum_{v=1}^{d-3} \sum_{t=l_{v-1}+1}^{l_v} A(l_1,\cdots,t-1,\cdots,l_{d-3},n-2) + \sum_{t=l_{d-3}+1}^{n-3} A(l_1,\cdots,l_{d-3},t) + 2A(l_1,\cdots,l_{d-3},n-2) = \alpha' + \beta' + 2\gamma'.$$

Note that $\beta' + \gamma' = \beta$. Hence

$$\begin{aligned} A(l,n) - \gamma &= A(l,n) - A(l,n-1) = \alpha + \beta + 2\gamma - \alpha' - \beta' - 2\gamma' \\ &= \alpha - \alpha' + 2\gamma - \gamma', \text{ so that} \end{aligned}$$

$$A(l,n) = \alpha - \alpha' + 3\gamma - \gamma' \leqq 3(\alpha+\gamma). \qquad \text{Q.E.D.}$$

In the rest of this section we shall use the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to estimate $A(l, n)$.

We need the following properties of the binomial coefficients:

a) It is well known that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

b) Let $0 \leqq n, l$ be any integers, then by induction on l it follows that $\sum_{k=0}^{l} \binom{n+k}{n} = \binom{n+l+1}{n+1}$. In particular, since $\binom{n+l}{n}$ appears as one of the summands on the left side, it follows that $\binom{n+l}{n} \leqq \binom{n+l+1}{n+1}$.

c) If $0 \leqq k, l, n$ and $k \leqq l$, then $\binom{n+k}{n} \leqq \binom{n+l}{n}$.

d) It is well known that $\binom{2n}{n} \leqq 4^n$ for all $0 \leqq n$.

Lemma 4.4. *Let $l_1, \cdots, l_r, n$ be a set of non negative integers, then:*

$$\sum_{v=1}^{r-1} \binom{n+l_1}{n} \cdots \binom{n+l_{v-1}}{n} \binom{n+l_v}{n+1} \binom{n+l_{v+1}}{n} \cdots \binom{n+l_r}{n}$$
$$+ \binom{n+l_1}{n} \cdots \binom{n+l_{r-1}}{n} \binom{n+l_r+1}{n+1}$$
$$\leqq \binom{n+l_1+1}{n+1} \cdots \binom{n+l_r+1}{n+1}$$

Proof. Denote the left side of the inequality by q. We prove the lemma by induction on r.

If $r = 1$, then, in fact, we have equality. Assume now the lemma is valid for $r - 1$. By using (a) and (b) of the previous remark we have:

$$q \leqq \left\{ \left[\sum_{v=1}^{r-1} \binom{n+l_1}{n} \cdots \binom{n+l_v}{n+1} \cdots \binom{n+l_{r-1}}{n} \right] \right.$$
$$\left. + \left[\binom{n+l_1}{n} \cdots \binom{n+l_{r-1}}{n} \right] \right\} \binom{n+l_r+1}{n+1}$$

$$= \left\{ \left[\sum_{v=1}^{r-2} \binom{n+l_1}{n} \cdots \binom{n+l_v}{n+1} \cdots \binom{n+l_{r-1}}{n} \right] \right.$$

$$+\left[\binom{n+l_1}{n}\cdots\binom{n+l_{r-2}}{n}\binom{n+l_{r-1}}{n+1}\right.$$

$$\left.\left.+\binom{n+l_1}{n}\cdots\binom{n+l_{r-1}}{n}\right]\right\}\binom{n+l_r+1}{n+1}$$

$$=\left\{\left(\sum_{v=1}^{r-2}\binom{n+l_1}{n}\cdots\binom{n+l_v}{n+1}\cdots\binom{n+l_{r-1}}{n}\right)\right]$$

$$\left.+\binom{n+l_1}{n}\cdots\binom{n+l_{r-2}}{n}\binom{n+l_{r-1}+1}{n+1}\right)\right\}\binom{n+l_r+1}{n+1}$$

$$\leqq\left\{\binom{n+l_1+1}{n+1}\quad\binom{n+l_{r-1}+1}{n+1}\right\}\binom{n+l_r+1}{n+1}\text{ by induction}$$

Q.E.D.

THEOREM 4.5. *Let* $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n$, *then*

$$A(l_1,\cdots,l_{d-3},n) \leqq 3^n\binom{n+l_1}{n}\cdots\binom{n+l_{d-3}}{n}$$

PROOF. By induction on n. If $n = 1$, then

$$A(l,1) = 1 < 3 \leqq 3^1\binom{1+l_1}{1}\cdots\binom{1+l_{d-3}}{1}$$

Assume that for any set of integers $s_1,\cdots,s_{d-3}$ such that $0 \leqq s_1 \leqq \cdots \leqq s_{d-3} \leqq n$, we have $A(s_1,\cdots,s_{d-3},n) \leqq 3^n\binom{n+s_1}{n}\cdots\binom{n+s_{d-3}}{n}$.

We show that under the induction assumption, if $0 \leqq l_1 \leqq \cdots \leqq l_{d-3} \leqq n+1$, then

$$A(l,n+1) \leqq 3^{n+1}\binom{n+1+l_1}{n+1}\cdots\binom{n+1+l_{d-3}}{n+1}.$$

Case I. Suppose $l_{d-3} \leqq (n+1)-2 = n-1$. Using Lemma 4.3 (with the substitution of $n+1$ for n) and the hypothesis of the induction, we have

$$A(l_1,\cdots,l_{d-3},n+1)$$

$$\leqq 3\left[\sum_{v=1}^{d-4}\sum_{t=l_{v-1}+1}^{l_v} A(l_1,\cdots,t-1,\cdots,l_{d-3},n) + \sum_{t=l_{d-4}+1}^{l_{d-3}+1} A(l_1,\cdots,l_{d-4},t-1,n)\right]$$

$$\leqq 3^{n+1}\left[\sum_{v=1}^{d-4}\sum_{t=l_{v-1}+1}^{l_v}\binom{n+l_1}{n}\cdots\binom{n+l_{v-1}}{n}\binom{n+t-1}{n}\binom{n+l_{v+1}}{n}\right.$$

$$\left.\cdots\binom{n+l_{d-3}}{n}\right]$$

$$+ 3^{n+1}\left[\sum_{t=l_{d-4}+1}^{l_{d-3}+1} \binom{n+l_1}{n} \cdots \binom{n+l_{d-4}}{n} \binom{n+t-1}{n}\right]$$

$$\leqq 3^{n+1}\left\{\sum_{\nu=1}^{d-4} \left[\sum_{t=1}^{l_\nu} \binom{n+t-1}{n}\right] \binom{n+l_1}{n} \cdots \binom{n+l_{\nu-1}}{n} \binom{n+l_{\nu+1}}{n} \cdots \binom{n+l_{d-3}}{n}\right\}$$

$$+ 3^{n+1}\left[\sum_{t=1}^{l_{d-3}+1} \binom{n+t-1}{n}\right] \binom{n+l_1}{n} \cdots \binom{n+l_{d-4}}{n}$$

$$= 3^{n+1}\left[\sum_{\nu=1}^{d-4} \binom{n+l_1}{n} \cdots \binom{n+l_{\nu-1}}{n} \binom{n+l_\nu}{n+1} \binom{n+l_{\nu+1}}{n} \cdots \binom{n+l_{d-3}}{n}\right]$$

$$+ 3^{n+1}\binom{n+l_1}{n} \cdots \binom{n+l_{d-4}}{n} \binom{n+l_{d-3}+1}{n+1}$$

$$\leqq 3^{n+1}\binom{n+l_1+1}{n+1} \cdots \binom{n+l_{d-3}+1}{n+1}$$

by property (b) of the binomial coefficients and by Lemma 4.4.

Case 2. If $n \leqq l_{d-3}$, then, according to Definition 4.1, $A(l, n+1) = A(h, n+1)$ where $h = h(l)$ satisfies $h_\nu \leqq l_\nu$ for all $1 \leqq \nu \leqq d-3$ and $h_{d-3} \leqq n-1$. Using Case I of this theorem and property (c) of the binomial coefficients we have:

$$A(l, n+1) = A(h, n+1) = A(h_1, \cdots, h_{d-3}, n+1)$$

$$\leqq 3^{n+1}\binom{n+h_1+1}{n+1} \cdots \binom{n+h_{d-3}+1}{n+1} \leqq 3^{n+1}\binom{n+l_1+1}{n+1} \cdots \binom{n+l_{d-3}+1}{n+1}$$

Q.E.D.

THEOREM 4.6. *Let d_n be the co-dimension of order n of the T-ideal $P^{(d)}$ generated by the identity $f(x_1, \cdots, x_d) = f_d(x_1, \cdots, x_d)$. Then, for every n, $d_n \leqq (3.4^{d-3})^n$.*

PROOF. It was noted ("Note" after Definition 1.5) that $d_n = a(n, \cdots, n, n)$. Using Lemma 4.2, Theorem 4.5 and property (d) of the binomial coefficient we have:

$$d_n = a(n, \cdots, n, n) \leqq A(n, \cdots, n, n)$$

$$\leqq 3^n \binom{n+n}{n} \cdots \binom{n+n}{n} = 3^n \cdot \binom{2n}{n}^{d-3} \leqq (3.4^{d-3})^n$$

Q.E.D.

THEOREM 4.7. *Let A be a PI-algebra and let $\{h_v\}$ be its sequence of co-dimensions. Assume that A satisfies a non trivial identity of degree d. Then, for all n, $h_n \leqq (3.4^{d-3})^n$.*

PROOF. Let Q be the T-ideal of identities of A in $F[x]$ and let $0 \neq f(x_1, \cdots, x_d) \in Q$ be a non trivial homogeneous multilinear identity for A. Let $P^{(d)}$ denote the T-ideal generated by $f(x_1, \cdots, x_d)$ (Definition 1.1), then it is clear that $P^{(d)} \subseteq Q$. Let $\{d_v\}$ be the sequence of co-dimensions of $P^{(d)}$. Then:

$$h_n = \dim_F \frac{V_n}{Q \cap V_n} \leqq \dim_F \frac{V_n}{P^{(d)} \cap V_n} = d_n.$$

Hence $h_n \leqq d_n$ for all n. Therefore, by Theorem 4.6, $h_n \leqq d_n \leqq (3.4^{d-3})^n$ Q.E.D.

5. Applications: existence of identities in $A \otimes B$

In this section we use Theorem 4.7 to show that the tensor product of two PI-algebras is again PI-algebra. We begin with two

REMARKS. (i) Let A, B be two PI-algebras over a field F. The elements $\{a_i \otimes b_i \mid a_i \in A,\ b_i \in B\}$ are linear generators of $A \otimes_F B$. Hence, if $g(x_1, \cdots, x_n)$ is a multilinear polynomial in $x_1, \cdots, x_n$, then $g(x)$ is an identity for $A \otimes_F B$ if and only if for any sets $a_1, \cdots, a_n \in A$ and $b_1, \cdots, b_n \in B$, $g(a_1 \otimes b_1, \cdots, a_n \otimes b_n) = 0$.

(ii) Let $Q \subseteq F[x]$ be the T-ideal of identities of a PI-algebra A, and let $\{d_v\}$ be its sequence of co-dimensions. If we write $Q \cap V_n = Q_n$ then, by Definition 1.4 we have: $d_n = \dim_F V_n / Q_n$. Now, the $n!$ monomials $\{x_{\sigma_1} \cdots x_{\sigma_n} \mid \sigma \in S_n\}$ span V_n, hence they generate also V_n modulo Q_n. Since the dimension of V_n modulo Q_n is d_n, there exist d_n monomials $M_1(x_1, \cdots, x_n), \cdots, M_{d_n}(x_1, \cdots, x_n)$ which form a basis for V_n over Q_n. Hence, for any monomial $x_{\sigma_1} \cdots x_{\sigma_n} \in V_n$ there exist $\phi_i(\sigma) \in F$, $i = 1, \cdots, d_n$, such that:

$$x_{\sigma_1} \cdots x_{\sigma_n} \equiv \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(x_1, \cdots, x_n) \pmod{Q_n}.$$

Since $Q_n \subseteq Q$, its elements are identities for A. It follows therefore that for any substitution $a_1, \cdots, a_n \in A$, we have the equality:

$$a_{\sigma_1} \cdots a_{\sigma_n} = \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a_1, \cdots, a_n), \qquad \sigma \in S_n.$$

With this preliminary we prove

THEOREM 5.1. *Let A, B be two PI-algebras over a field F, then $A \otimes_F B$ is a PI-algebra.*

PROOF. Let $\{d_\nu\}$ be the sequence of co-dimensions of A, $\{h_\nu\}$ that of B. By Theorem 4.7 there exist real positive numbers k, l such that for all n, $d_n \leqq k^n$ and $h_n \leqq l^n$. It is well known that there exist n such that $k^n \cdot l^n < n!$, hence $d_n \cdot h_n < n!$.

We prove that for this n, $A \otimes_F B$ satisfies a non trivial multilinear homogeneous identity of degree n.

Let $M_1(x_1, \cdots, x_n), \cdots, M_{d_n}(x_1, \cdots, x_n)$ be d_n monomials in $x_1, \cdots, x_n$, and $\phi_i(\sigma) \in F$, $1 \leqq i \leqq d_n$, $\sigma \in S_n$, such that for any substitution $a_1, \cdots, a_n \in A$ and $\sigma \in S_n$ we have $a_{\sigma_1} \cdots a_{\sigma_n} = \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a_1, \cdots, a_n)$. (See the previous remark, (ii)).

Similarly, let $N_1(x_1, \cdots, x_n), \cdots, N_{h_n}(x_1, \cdots, x_n)$ and $\psi_j(\sigma) \in F$, $1 \leqq j \leqq h_n$, $\sigma \in S_n$ be monomials and coefficients such that for any substitution $b_1, \cdots, b_n \in B$ and $\sigma \in S_n$ we have: $b_{\sigma_1} \cdots b_{\sigma_n} = \sum_{j=1}^{h_n} \psi_j(\sigma) N_j\ (b_1, \cdots, b_n)$.

Let $g(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma_1} \cdots x_{\sigma_n}$ be any multilinear polynomial with arbitrary coefficients $\{\alpha_\sigma\}$. Let $a_1, \cdots, a_n \in A$, $b_1, \cdots, b_n \in B$. Write

$$
\begin{aligned}
(*) \cdots \quad & g(a_1 \otimes b_1, \cdots, a_n \otimes b_n) \\
&= \sum_{\sigma \in S_n} \alpha_\sigma (a_{\sigma_1} \otimes b_{\sigma_1}) \cdots (a_{\sigma_n} \otimes b_{\sigma_n}) \\
&= \sum_{\sigma \in S_n} \alpha_\sigma (a_{\sigma_1} \cdots a_{\sigma_n}) \otimes (b_{\sigma_1} \cdots b_{\sigma_n}) \\
&= \sum_{\sigma \in S_n} \alpha_\sigma \left(\sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a) \right) \otimes \left(\sum_{j=1}^{h_n} \psi_j(\sigma) N_j(b) \right) \\
&= \sum_{i=1}^{d_n} \sum_{j=1}^{h_n} \left(\sum_{\sigma \in S_n} \phi_i(\sigma) \psi_j(\sigma) \alpha_\sigma \right) M_i(a) \otimes N_j(b).
\end{aligned}
$$

Consider the system of homogeneous linear equations $\sum_{\sigma \in S_n} \phi_i(\sigma) \psi_j(\sigma) \alpha_\sigma = 0$ for all i, j. This is a set of $d_n \cdot h_n$ equations with coefficients $\phi_i(\sigma)\psi_j(\sigma)$ and $n!$ unknown indeterminates α_σ's. Since $d_n \cdot h_n < n!$, there exists a non trivial solution $\{\alpha_\sigma\}$ for this system. Clearly, (*) implies that $g(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma_1} \cdots x_{\sigma_n}$ is a non trivial identity for $A \otimes_F B$.

Q.E.D.

COROLLARY 5.2. *Let $0 < d, h$ be any integers, and let $n = n(d, h)$ be the minimal integer such that $(3.4^{d-3} \cdot 3.4^{n-3})^n < n!$*

Let A, B be two PI-algebras which, respectively, satisfy minimal identities of degree d and h. It follows from Theorem 4.7 and from the proof of Theorem 5.1, *that* $A \otimes_F B$ *satisfies an identity of degree* n. *This integer* $n = n(d,h)$ *is independent of the special algebras* A *and* B, *and depend only on the degrees of their minimal identities* d *and* h.

Added in proof: By the same methods the following (more general) theorem can be proved.

THEOREM 1. *Let* R *be a commutative ring with an identity element, and let* A, B *be two* R*-algebras satisfying proper identities over* R, *then* $A \otimes_R B$ *satisfies a proper identity over* R.

Amitsur [*] has shown that if A satisfies a proper identity over R, then it also satisfies a multilinear identity with coefficients ± 1 and 0. Thus A satisfies a proper identity over Z (the integers).

Replace F by Z, $F[x]$ by $Z[x]$, $sp_F\{\cdots\}$ by $Sp_z\ \{\cdots\}$ etc. For instance, $V_n(x) = Sp_z\{x_{\sigma_1} \cdots x_{\sigma_n} | \sigma \in S_n\}$ and $Q \subseteq Z[x]$ is the T-ideal of identities of A over Z. There exist an integer r, $1 \leqq r \leqq n!$, and monomials $M_1(x), \cdots, M_r(x) \in V_n(x)$ such that $M_1, \cdots, M_r$ span (over Z) $V_n(x)$ modulo Q. Let d_n be the minimal such r. $\{d_v\}$ is the sequence of codimensions of A (over Z).

Now, the whole paper can be read again with these new notations to get the following result: for all n, $d_n \leqq (3.4^{d+3})^n$.

We can now prove Theorem 1:

This is done by following the proof of Theorem 5.1 until we get the system of homogeneous linear equations

$$\sum_{\sigma \in S_n} \phi(\sigma_1)\psi_i(\sigma)\alpha_\sigma = 0,$$

now with integral coefficients. A reduced integral solution (which obviously exists) for the system gives us the desired proper identity.

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